Note on star-triangle equivalence in conducting networks

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Abstract

By using the discrete Poisson equations the star-triangle (external) equivalence in conducting networks is considered and the Kennelly famous transformation formulae [Kennelly A E 1899 Electrical World and Engineer **34** 413] are explicitly restated.

1 Introduction and outline of the paper

The homological representation and modeling [1] of networks (n/w) is based on their geometric elements, called also the chains – nodes, branches (edges), meshes (simple closed loops), and using the natural geometric boundary operator of the n/w which only depends on the geometry (topology) of the n/w. Then, both of the Kirchhoff laws can be presented in a compact algebraic form that may be called the homological Kirchhoff Laws.

In the present note, we compose the discrete Poisson equations and consider the star-triangle (external) equivalence transformation in conducting networks, see Fig. 1.1, and prove the Kennelly famous transformation formulae [2]. We use the geometrical representation that was explained in [3].

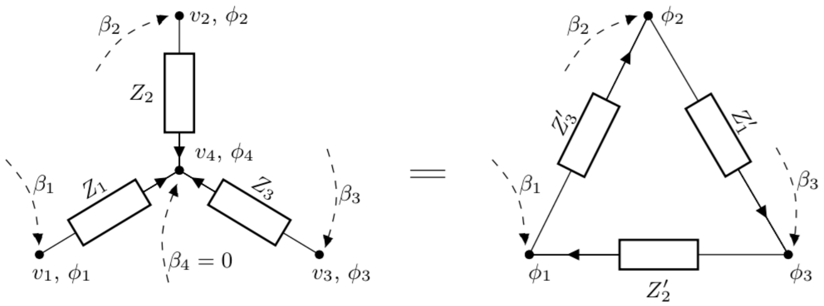

Figure 1.1: Star-triangle (external/boundary) equivalence

First introduce some notations, here we follow [3]. The both circuits are assumed to have the same boundary conditions. We denote

$$|\beta\rangle := |\beta_1\beta_2\beta_3\beta_4\rangle\,, \quad |\beta\rangle' := |\beta_1\beta_2\beta_3\rangle \quad \text{(boundary currents)} \tag{1.1a}$$

$$|\phi\rangle := |\phi_1\phi_2\phi_3\phi_4\rangle\,, \quad |\phi\rangle' := |\phi_1\phi_2\phi_3\rangle \quad \text{(node potentials)} \tag{1.1b}$$

The impedance matrices are

$$Z := \begin{bmatrix} Z_1 & 0 & 0 \\ 0 & Z_2 & 0 \\ 0 & 0 & Z_3 \end{bmatrix}, \quad Z' := \begin{bmatrix} Z'_3 & 0 & 0 \\ 0 & Z'_1 & 0 \\ 0 & 0 & Z'_2 \end{bmatrix} \tag{1.2}$$

The admittances Y_n and Y'_n are defined by

$$Y_n Z_n = 1 = Y'_n Z'_n, \quad n = 1,2,3 \tag{1.3}$$

and the admittance matrices are

$$Y := Z^{-1} = \begin{bmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Y_3 \end{bmatrix}, \quad Y' := Z'^{-1} = \begin{bmatrix} Y'_3 & 0 & 0 \\ 0 & Y'_1 & 0 \\ 0 & 0 & Y'_2 \end{bmatrix} \tag{1.4}$$

2 Star

Consider the star circuit represented on Fig. 1.1. Define

- *Node space* $C_0 := \langle v_1 v_2 v_3 v_4 \rangle_{\mathbb{C}}, \quad \dim C_0 = 4$
- *Branch space* $C_1 := \langle e_1 e_2 e_3 \rangle_{\mathbb{C}}, \quad \dim C_1 = 3$

First construct the boundary operator $\partial : C_1 \to C_0$. By definition,

$$\partial e_1 = \partial(v_1 v_4) := v_4 - v_1 =: |-1;0;0;1\rangle \tag{2.1a}$$
$$\partial e_2 = \partial(v_2 v_4) := v_4 - v_2 =: |0;-1;0;1\rangle \tag{2.1b}$$
$$\partial e_3 = \partial(v_3 v_4) := v_4 - v_3 =: |0;0;-1;1\rangle \tag{2.1c}$$

and in the matrix representation we have

$$\partial = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \partial^T = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{2.2}$$

Now it is easy to calculate the Laplacian as follows:

$$\Delta := \partial Y \partial^T \tag{2.3a}$$
$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Y_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{2.3b}$$
$$= \begin{bmatrix} Y_1 & 0 & 0 & -Y_1 \\ 0 & Y_2 & 0 & -Y_2 \\ 0 & 0 & Y_3 & -Y_3 \\ -Y_1 & -Y_2 & -Y_3 & Y_1 + Y_2 + Y_3 \end{bmatrix} \tag{2.3c}$$

The Poisson equation

$$\Delta |\phi\rangle = - |\beta\rangle \tag{2.4}$$

in coordinate form reads

$$\begin{cases} \beta_1 = \dfrac{-\phi_1 + \phi_4}{Z_1} \\ \beta_2 = \dfrac{-\phi_2 + \phi_4}{Z_2} \\ \beta_3 = \dfrac{-\phi_3 + \phi_4}{Z_3} \\ \beta_4 = -\left(\dfrac{-\phi_1 + \phi_4}{Z_1} + \dfrac{-\phi_2 + \phi_4}{Z_2} + \dfrac{-\phi_3 + \phi_4}{Z_3} \right) \end{cases} \tag{2.5}$$

We can easily check consistency:

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0 \tag{2.6}$$

For $\beta_4 = 0$ we have

$$\frac{-\phi_1 + \phi_4}{Z_1} + \frac{-\phi_2 + \phi_4}{Z_2} + \frac{-\phi_3 + \phi_4}{Z_3} = 0 \tag{2.7}$$

from which it follows that

$$\phi_4 = \frac{\phi_1 Y_1 + \phi_2 Y_2 + \phi_3 Y_3}{Y_1 + Y_2 + Y_3} \tag{2.8}$$

3 Triangle

Next consider the the triangle circuit on Fig. 1.1. We denote the spanning nodes and branches by the same letters. Then the linear spans are

- *Node space* $C_0 := \langle v_1 v_2 v_3 \rangle_{\mathbb{C}}\,, \quad \dim C_0 = 3$
- *Branch space* $C_1 := \langle e_1 e_2 e_3 \rangle_{\mathbb{C}}\,, \quad \dim C_1 = 3$

Construct the boundary operator $\partial : C_1 \to C_0$. We can see that

$$\partial e_1 = \partial(v_1 v_2) := v_2 - v_1 =: |-1; 1; 0\rangle \tag{3.1}$$

$$\partial e_2 = \partial(v_2 v_3) := v_3 - v_2 =: |0; -1; 1\rangle \tag{3.2}$$

$$\partial e_3 = \partial(v_3 v_1) := v_1 - v_3 =: |1; 0; -1\rangle \tag{3.3}$$

and the matrix representation is

$$\partial = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \Longrightarrow \quad \partial^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \tag{3.4}$$

The Laplacian is

$$\Delta' := \partial Y' \partial^T \tag{3.5a}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} Y'_3 & 0 & 0 \\ 0 & Y'_1 & 0 \\ 0 & 0 & Y'_2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \tag{3.5b}$$

$$= \begin{bmatrix} Y'_3 + Y'_2 & -Y'_3 & -Y_2 \\ -Y'_3 & Y'_3 + Y'_1 & -Y'_1 \\ -Y'_2 & -Y'_1 & Y'_1 + Y'_2 \end{bmatrix} \tag{3.5c}$$

The Poisson equation is

$$\Delta' \,|\phi\rangle' = -\,|\beta\rangle' \tag{3.6}$$

Hence we have

$$\begin{cases} \beta_1 = \dfrac{-\phi_1 + \phi_2}{Z'_3} - \dfrac{\phi_1 - \phi_3}{Z'_2} \\[2ex] \beta_2 = -\dfrac{-\phi_1 + \phi_2}{Z'_3} + \dfrac{-\phi_2 + \phi_3}{Z'_1} \\[2ex] \beta_3 = -\dfrac{-\phi_2 + \phi_3}{Z'_1} + \dfrac{\phi_1 - \phi_3}{Z'_2} \end{cases} \tag{3.7}$$

Check the consistency:

$$\beta_1 + \beta_2 + \beta_3 = 0 \tag{3.8}$$

4 Equivalence

Now consider the star-triangle equivalence as exposed on Fig. 1.1 and prove the Kennelly theorem.

Theorem 4.1 (A. E. Kennelly [2]). *If the (external/boundary) equivalence presented on Fig. 1.1 holds, then one has*

$$\boxed{Z_n Z'_n = Z'_1 Z'_2 Z'_3/(Z'_1 + Z'_2 + Z'_3) = Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1, \quad n = 1, 2, 3} \tag{4.1}$$

Proof. As soon as the boundary currents on Fig. 1.1 are considered the same, then we have

$$\frac{-\phi_1 + \phi_4}{Z_1} = \frac{-\phi_1 + \phi_2}{Z'_3} - \frac{\phi_1 - \phi_3}{Z'_2} \quad | \cdot Z_1 \tag{4.2a}$$
$$\frac{-\phi_2 + \phi_4}{Z_2} = -\frac{-\phi_1 + \phi_2}{Z'_3} + \frac{-\phi_2 + \phi_3}{Z'_1} \quad | \cdot Z_2 \tag{4.2b}$$
$$\frac{-\phi_3 + \phi_4}{Z_3} = -\frac{-\phi_2 + \phi_3}{Z'_1} + \frac{\phi_1 - \phi_3}{Z'_2} \quad | \cdot Z_3 \tag{4.2c}$$

where ϕ_4 is given by (2.8). Hence we obtain equations for the potentials ϕ_1, ϕ_2, ϕ_3,

$$-\phi_1 + \phi_4 = (-\phi_1 + \phi_2)\frac{Z_1}{Z'_3} - (\phi_1 - \phi_3)\frac{Z_1}{Z'_2} \tag{4.3a}$$
$$-\phi_2 + \phi_4 = -(-\phi_1 + \phi_2)\frac{Z_2}{Z'_3} + (-\phi_2 + \phi_3)\frac{Z_2}{Z'_1} \tag{4.3b}$$
$$-\phi_3 + \phi_4 = -(-\phi_2 + \phi_3)\frac{Z_3}{Z'_1} + (\phi_1 - \phi_3)\frac{Z_3}{Z'_2} \tag{4.3c}$$

By eliminating here the potential ϕ_4, we get relations for the boundary potentials,

$$-\phi_1 + \phi_2 = (-\phi_1 + \phi_2)\frac{Z_1}{Z'_3} - (\phi_1 - \phi_3)\frac{Z_1}{Z'_2} + (-\phi_1 + \phi_2)\frac{Z_2}{Z'_3} - (-\phi_2 + \phi_3)\frac{Z_2}{Z'_1} \tag{4.4a}$$
$$-\phi_2 + \phi_3 = -(-\phi_1 + \phi_2)\frac{Z_2}{Z'_3} + (-\phi_2 + \phi_3)\frac{Z_2}{Z'_1} + (-\phi_2 + \phi_3)\frac{Z_3}{Z'_1} - (\phi_1 - \phi_3)\frac{Z_3}{Z'_2} \tag{4.4b}$$
$$-\phi_3 + \phi_1 = -(-\phi_2 + \phi_3)\frac{Z_3}{Z'_1} + (\phi_1 - \phi_3)\frac{Z_3}{Z'_2} - (-\phi_1 + \phi_2)\frac{Z_1}{Z'_3} + (-\phi_1 + \phi_3)\frac{Z_1}{Z'_2} \tag{4.4c}$$

We can easily check consistency of the last Eqs, by summing these we easily obtain $0 = 0$. This means that one equation is a linear combination of others and we can variate the independent potentials ϕ_1, ϕ_1, ϕ_3 only in two equations. We use the first two Eqs.

By variating the independent potentials ϕ_1, ϕ_1, ϕ_3 and setting the nontrivial potential $\phi_3 = 1$ in the first equation we obtain

$$0 = \frac{Z_1}{Z'_2} - \frac{Z_2}{Z'_1} \quad \Longrightarrow \quad \boxed{Z_1 Z'_1 = Z_2 Z'_2} \tag{4.5}$$

Now take $\phi_1 = 1$,

$$1 = \frac{Z_1}{Z'_3} + \frac{Z_1}{Z'_2} + \frac{Z_2}{Z'_3} \quad \Longrightarrow \quad 1 = \frac{Z_1 Z'_2 + Z_1 Z'_3 + Z_2 Z'_2}{Z'_2 Z'_3} \tag{4.6a}$$
$$= \frac{Z_1 Z'_2 + Z_1 Z'_3 + Z_1 Z'_1}{Z'_2 Z'_3} \tag{4.6b}$$

$$= \frac{Z_1(Z'_2 + Z'_3 + Z'_1)}{Z'_2 Z'_3} \quad \Longrightarrow \quad \boxed{Z_1 = \frac{Z'_2 Z'_3}{Z'_2 + Z'_3 + Z'_1}} \tag{4.6c}$$

Next take $\phi_2 = 1$,

$$1 = \frac{Z_1}{Z'_3} + \frac{Z_2}{Z'_3} + \frac{Z_2}{Z'_1} \quad \Longrightarrow \quad 1 = \frac{Z_1 Z'_1 + Z_2 Z'_1 + Z_2 Z'_3}{Z'_3 Z'_1} \tag{4.7a}$$

$$= \frac{Z_2 Z'_2 + Z_2 Z'_1 + Z_2 Z'_3}{Z'_3 Z'_1} \tag{4.7b}$$

$$= \frac{Z_2(Z'_2 + Z'_1 + Z'_3)}{Z'_3 Z'_1} \quad \Longrightarrow \quad \boxed{Z_2 = \frac{Z'_1 Z'_3}{Z'_2 + Z'_1 + Z'_3}} \tag{4.7c}$$

By variating the independent potentials ϕ_1, ϕ_1, ϕ_3 in the second equation and setting there the nontrivial potential $\phi_1 = 1$, we obtain

$$0 = \frac{Z_2}{Z'_3} - \frac{Z_3}{Z'_2} \quad \Longrightarrow \quad \boxed{Z_2 Z'_2 = Z_3 Z'_3} \tag{4.8}$$

By setting $\phi_3 = 1$, we obtain

$$1 = \frac{Z_2}{Z'_1} + \frac{Z_3}{Z'_1} + \frac{Z_3}{Z'_2} \quad \Longrightarrow \quad 1 = \frac{Z_2 Z'_2 + Z_3 Z'_2 + Z_3 Z'_1}{Z'_1 Z'_2} \tag{4.9a}$$

$$= \frac{Z_3 Z'_3 + Z_3 Z'_2 + Z_3 Z'_1}{Z'_1 Z'_2} \tag{4.9b}$$

$$= \frac{Z_3(Z'_3 + Z'_2 + Z'_1)}{Z'_1 Z'_2} \quad \Longrightarrow \quad \boxed{Z_3 = \frac{Z'_1 Z'_2}{Z'_3 + Z'_2 + Z'_1}} \tag{4.9c}$$

One can easily check that other variations of the potentials ϕ_n $(n = 1, 2, 3)$ do not produce additional constraints. □

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References

[1] Roth J P 1955 *Proc. Nat. Acad. Sci. USA* **41** 518
[2] Kennelly A E 1899 Electrical World and Engineer **34** 413
[3] Paal E and Umbleja M 2014 *J. Phys.: Conf. Series* **532** 012022

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